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A ROBUST ESTIMATOR OF THE DIFFERENCE BETWEEN LOCATION PARAMETERS OF CORRELATED VARIABLES FOR FRAGMENTARY SAMPLES.*

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A simple and robust estimator of the difference of location parameters of correlated variables is proposed when some observations on either of the variables are missing. We show that this estimator is consistent, asymptotically normally distributed, and insensitive to outlying observations. Asymptotic relative efficiency comparisons with other known estimators are made to show the advantage of the proposed estimator.

A ROBUST ESTIMATOR OF THE DIFFERENCE BETWEEN LOCATION PARAMETERS

OF CORRELATED VARIABLES FOR FRAGMENTARY SAMPLES

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A simple and robust estimator of the difference of location parameters of correlated variables is proposed when some observations on either of the variables are missing. We show that this estimator is consistent, asymptotically normally distributed, and insensitive to outlying observations. Asymptotic relative efficiency comparisons with other known estimators are made to show the advantage of the proposed estimator.

Key words and phrases: **Consistent, Median, Asymptotic relative efficiency, Bivariate logistic distribution, Bivariate exponential distribution.**

1. INTRODUCTION

The problem of estimating the difference between means of a bivariate normal distribution when some observations on either of the variables are missing has received a great deal of attention in recent statistical literature (c.f. Wilks (1932), Anderson (1957), Hocking and Smith (1968), Mehta & Gurland (1969), Lin (1971, 1973) and Lin & Stivers (1974)). In this article we study the problem of estimation of the difference between the location parameters of correlated variables from fragmentary samples when the population being sampled is not necessarily normal. More specifically, let $(X,Y-\theta)'$ be a random vector with absolutely continuous joint distribution function H which is free of θ and is symmetric in its arguments, i.e. $H(u,v) = H(v,u)$, for $(u,v) \in R^2$. The marginal distribution function of X is denoted by F and its density function f is assumed to satisfy the condition $\int f^2(x)dx < \infty$. Now, let $(X_i,Y_i)'$, $i = 1,\cdots,n$ be n pairs of observations on $(X,Y)'$; X_{n+j}, $j = 1,\cdots,s$, be s additional observations on X; Y_{n+k}, $k = 1,\cdots,t$, be t additional observations on Y. The $(X_i,Y_i)'$, X_{n+j}, and Y_{n+k} are assumed to be mutually independent for $i = 1,\cdots,n$, $j = 1,\cdots,s$ and $k = 1,\cdots,t$. The problem is how to use the fragmentary samples in the most efficient way to estimate the shift parameter θ. Gupta and Rohatgi (1979) considered the case that X and Y are linearly related and constructed regression estimators which are linear combinations of fragmentary sample means. Therefore, their estimators are sensitive to outlying observations.

A simple and robust estimator $\hat{\theta}_0$ of θ is proposed in Section 2. We show that this estimator is unbiased if the underlying distribution H is symmetric about some point $(\mu_1,\mu_2)'$ or the two fragmentary sample sizes are equal, i.e., $s = t$. Also, it is shown that $\hat{\theta}_0$ is consistent and asymptotically normally distributed. In Section 3 we compare the asymptotic relative efficiency of $\hat{\theta}_0$ with other known estimators.

An outstanding feature of $\hat{\theta}_0$ is that it can be used even when the pairing of X_i with Y_i, $i = 1, \cdots, n$, cannot be identified (c.f. Hollander, Pledger, and Lin (1974)). For example, a statewide readiness test was given at the beginning of the 1979-80 school year to every incoming first grade public school student of South Carolina. The purpose of this test was to distinguish those students who were ready for the formal first grade curriculum from those who were not ready. A pilot testing was conducted to obtain the cutoff score using a random sample of South Carolina's kindergarten students at the end of the 1978-79 school year (Garcia-Quintana & Huynh, 1980). Educators have constantly demonstrated that in the very early years of schooling a vast amount of a student's achievement is caused by maturation and not necessarily by instruction. This coupled with the fact that these two tests were conducted approximately four months apart establishes a concern as to how much the cutoff score previously determined by the pilot test should be moved upward. So the problem becomes estimating "maturational growth" occurred during the summer months. However, many of the students in the pilot test cannot be identified at the data analysis time due to various human factors. Therefore, all the parametric and regression estimation procedures mentioned above are not valid. A similar example was also cited by Hollander, Pledger and Lin (1974).

2. THE ESTIMATOR $\hat{\theta}_0$

For the fragmentary samples given in Section 1, denote the ordered set of $(n+s)(n+t)$ differences $Y_j - X_i$ by $D_{(1)} < D_{(2)} < \cdots < D_{((n+s)(n+t))}$. A natural estimator $\hat{\theta}_0$ of θ is then the median of the D's, i.e.,

$$\hat{\theta}_0 = \begin{cases} (D_{(\ell)} + D_{(\ell+1)})/2 , & (n+s)(n+t) = 2\ell , \\ D_{(\ell+1)} , & (n+s)(n+t) = 2\ell + 1 . \end{cases}$$

First, we consider some exact small sample properties of $\hat{\theta}_0$.

Theorem 1. The distribution of the difference $\hat{\theta}_0 - \theta$ is free of θ .

Proof. See Appendix.

Theorem 2. The estimator $\hat{\theta}_0$ is distributed symmetrically about θ if either of the following two conditions hold:

(a) The distribution H is symmetric about some point $(\mu_1, \mu_2)'$.

(b) The two fragmentary sample sizes are equal, i.e., $s = t$.

Proof. See Appendix.

This theorem shows that under the stated conditions, the estimator $\hat{\theta}_0$ is unbiased. Now, let us consider the asymptotic performance of $\hat{\theta}_0$.

Theorem 3. Let $N = 2n + s + t$ and $\lambda_1, \lambda_2, \lambda_3$ be nonnegative numbers such that $\lambda_1 + \lambda_3 > 0$, $\lambda_2 + \lambda_3 > 0$ and $\lambda_1 + \lambda_2 + 2\lambda_3 = 1$. Then, as $N \to \infty$, $s/N \to \lambda_1$, $t/N \to \lambda_2$, $n/N \to \lambda_3$, the distribution of $N^{\frac{1}{2}}(\hat{\theta}_0 - \theta)$ converges to a normal distribution with mean 0 and variance

$$\sigma^2_{\hat{\theta}_0} = \left[\frac{1}{12} + \lambda_3\left(\frac{1}{2} - 2\int F(u)F(v)\,dH(u,v)\right)\right] \Big/ \left[(\lambda_3 + \lambda_1)(\lambda_3 + \lambda_2)\left(\int f^2(v)\,dv\right)^2\right] .$$

Proof. See Appendix.

The above theorem also shows that $\hat{\theta}_0$ is consistent. We note that the computation of $\hat{\theta}_0$ requires finding the median of the $((n+s)(n+t))$ differences

$Y_j - X_i$ and becomes rather tedious for a large set of data. Fortunately, there are several shortcut methods of obtaining $\hat{\theta}_0$ (c.f. Lehmann (1975), p. 83).

3. THE EFFICIENCY OF $\hat{\theta}_0$

The estimator $\hat{\theta}_0$ is compared with the Lin-Stivers estimator $\hat{\theta}_1$ (c.f. (2.4) of Lin & Stivers (1974)), a regression estimator $\hat{\theta}_2$ (c.f. (9) of Gupta & Rohatgi (1979)) and a naive estimator $\hat{\theta}_3 = \overline{Y} - \overline{X}$, where $\overline{Y} = \sum_{i=1}^{n+t} Y_i/(n+t)$ and $\overline{X} = \sum_{i=1}^{n+s} X_i/(n+s)$. All the estimators mentioned above are consistent and asymptotically normally distributed.

The relative efficiency of two estimators $\hat{\theta} = \hat{\theta}(N)$ and $\hat{\theta}' = \hat{\theta}'(N')$ of θ is defined as the ratio of sample sizes N and N' required for the estimators to have the same probability of falling within a stated distance of θ, i.e., for a fixed constant b,

$$P_\theta(|\hat{\theta}(N) - \theta| < b) = P_\theta(|\hat{\theta}'(N') - \theta| < b) . \tag{3.1}$$

However, this approach creates a problem in the large sample case. If $\hat{\theta}$ and $\hat{\theta}'$ are consistent estimators of θ, then the limiting probability in (3.1) of both estimators is 1. Thus it is impossible to use this probability to provide a meaningful comparison between $\hat{\theta}$ and $\hat{\theta}'$ asymptotically. To overcome this difficulty, we propose the following definition of the asymptotic relative efficiency of estimators.

Definition 1. Let $\{\hat{\theta}(N_k)\}$ and $\{\hat{\theta}'(N_k')\}$ be two sequences of estimators for estimating θ. Let $\{\theta \pm b_k\}$ be a sequence of intervals such that $\lim_{k \to \infty} b_k = 0$. In addition, let $\beta_{\hat{\theta}}(k) = P_\theta(|\hat{\theta}(N_k) - \theta| < b_k)$ and $\beta_{\hat{\theta}'}(k) = P_\theta(|\hat{\theta}'(N_k') - \theta| < b_k)$. Also, let $\{N_k\}$ and $\{N_k'\}$ be increasing sequences of positive integers such that the two sequences of estimators have the same limiting β value, i.e., $\lim_{k \to \infty} \beta_{\hat{\theta}}(k) = \lim_{k \to \infty} \beta_{\hat{\theta}'}(k)$. Then the asymptotical relative efficiency (ARE) of $\{\hat{\theta}\}$ and $\{\hat{\theta}'\}$ is

$$\mathrm{ARE}(\hat{\theta},\hat{\theta}') = \lim_{k \to \infty} (N_k'/N_k) ,$$

provided that this limit is the same for all such sequences $\{N_k\}$ and $\{N_k'\}$ and independent of the $\{\theta \pm b_k\}$ sequence.

Although there is no explicit statement concerning the rate of convergence of the sequence $\{b_k\}$ in the above definition, we are particularly interested in the sequence $\{\theta \pm b_k\}$ with $b_k = a/\sqrt{N_k} = a'/\sqrt{N_k'}$, where a and a' are some fixed constants. If both random variables $\sqrt{N_k}(\hat{\theta}-\theta)$ and $\sqrt{N_k'}(\hat{\theta}'-\theta)$ are asymptotically normally distributed with mean 0 and variances $\sigma_{\hat{\theta}}^2$ and $\sigma_{\hat{\theta}'}^2$, respectively, then it can be shown that $\mathrm{ARE}(\hat{\theta},\hat{\theta}') = \sigma_{\hat{\theta}'}^2/\sigma_{\hat{\theta}}^2$ which is the conventional definition of ARE of estimators $\hat{\theta}$ and $\hat{\theta}'$ provided that $\lim_{k \to \infty} \sqrt{N_k}(\hat{\theta}-\theta) = \sigma_{\hat{\theta}}^2$ and $\lim_{k \to \infty} \sqrt{N_k'}(\hat{\theta}'-\theta) = \sigma_{\hat{\theta}'}^2$ (c.f. Randles and Wolfe (1979), p. 227).

Several bivariate distributions were considered for H in the comparison study:

(a) A bivariate normal distribution with unit variances and correlation coefficient ρ .

(b) A bivariate logistic distribution (Gumbel (1961)). The joint distribution function is

$$H(u,v) = F(u)F(v)\{1+\alpha(1-F(u))(1-F(v))\} ,$$

where $-1 \le \alpha \le 1$, $F(u) = (1+e^{-u})^{-1}$, and the correlation coefficient $\rho = 3\alpha/\pi^2$.

(c) A bivariate exponential distribution (Gumbel (1960)). The joint distribution function is

$$H(u,v) = F(u)F(v)\left\{1+\alpha\left(1-F(u)\right)\left(1-F(v)\right)\right\} \text{ , where}$$

$F(u) = 1-e^{-u}$ and the correlation coefficient $\rho = \alpha/4$.

The advantage of using Gumbel's bivariate distributions for our comparison study is that the asymptotic variance $\sigma_{\hat\theta_0}^2$ of $\hat\theta_0$ in Theorem 3 has a closed form. For each family of bivariate distributions mentioned above and a group of selected values of ρ , the asymptotic relative efficiencies e_i = $ARE(\hat\theta_0,\hat\theta_i)$, $i = 1,2,3,$ were computed. Table 1 provides the computational results for $\lambda_1 = .1$, $\lambda_2 = .1$ and $\lambda_3 = .4$. Compared with $\hat\theta_i$, $i = 1,2,3,$ $\hat\theta_0$ does much better for heavy tail distributions as is to be expected because $\hat\theta_i$ $(i = 1,2,3)$ is a linear combination of fragmentary sample means. In particular, if H is a bivariate Cauchy distribution (c.f. Johnson & Kotz (1976), p. 295) which is not shown in Table 1, the $ARE(\hat\theta_0,\hat\theta_i) = \infty$, $i = 1,2,3$. For normal and logistic distributions, $\hat\theta_0$ performs as well as $\hat\theta_i$ $(i = 1,2,3)$ except for large values of ρ .

Results similar to Table 1 were also found for other combinations of λ_i , $i = 1,2,3$. Here, we only report a case when there is a large fraction of missing observations in the data (see Table 2).

4. APPENDIX

Proof of Theorem 1. The error of estimator $\hat\theta_0 - \theta = \text{med}\,(Y_j - X_i) - \theta$ can also be written as $\text{med}(Y_j-\theta-X_i)$. Since the joint distribution of $(X_i,Y_i-\theta)'$, X_{n+j} , and $Y_{n+k}-\theta$, $i = 1,\cdots,n$, $j = 1,\cdots,s$, $k = 1,\cdots,t$ is free of θ , the same is true for the distribution of $\hat\theta_0 - \theta$.

Proof of Theorem 2. By Lemma 1, for any real number a, $P_\theta(\hat\theta_0-\theta < a) = P_0(\hat\theta_0 < a)$. Without loss of generality, we can assume that $\theta = 0$.

(a) Since H is symmetric in its arguments, $\mu_1 = \mu_2 = \mu$. Also, since $\hat\theta_0 = \mathrm{med}(Y_j-X_i)$, we can assume that $\mu = 0$. It follows that $(X_i,Y_i)'$, X_{n+j}, and Y_{n+k} have the same joint distribution as $(-X_i,-Y_i)'$, $-X_{n+j}$, and $-Y_{n+k}$, $i = 1,\cdots,n$, $j = 1,\cdots,s$, $k = 1,\cdots,t$. This implies that $\hat\theta_0$ and $-\hat\theta_0$ have the same distribution and $\hat\theta_0$ is symmetric about 0.

(b) If $s = t$ and $\theta = 0$, then (X_i,Y_i), X_{n+j}, and Y_{n+k} have the same joint distribution as (Y_i,X_i), Y_{n+j}, and X_{n+k}, $i = 1,\cdots,n$, $j,k = 1,\cdots,s$. Hence, $\hat\theta_0$ and $-\hat\theta_0$ have the same distribution.

Two lemmas are needed to prove Theorem 3. First, let us define a scoring function ϕ for comparing two observations X_i and Y_j by

$$\phi(X_i,Y_j) = \begin{cases} 1, & X_i < Y_j, \\ 0, & \text{otherwise}, \end{cases}$$

and $W_{X,Y} = \sum_{i=1}^{n+s}\sum_{j=1}^{n+t}\phi(X_i,Y_j)$.

Lemma 1. For any real number a and integer i between 1 and $(n+s)(n+t)$, the i^{th} ordered difference $D_{(i)} \le a$ if and only if $W_{X,Y-a} \le (n+s)(n+t) - i$.

Proof. C. f. Theorem 4 of Chapter 2, p. 87, of Lehmann (1975).

Lemma 2. For $\theta = 0$ and a positive real a, the distribution of $\left((n+s)(n+t)N\right)^{-1/2}\left(W_{X,Y-a/\sqrt{N}} - (n+s)(n+t)p_1\right)$ converges to a normal distribution with mean 0 and variance $\eta^2 = 1/12 + \lambda_3\left(\tfrac{1}{2} - 2\int F(u)F(v)\,dH(u,v)\right)$, as $N \to \infty$,

$s/N \to \lambda_1$, $t/N \to \lambda_2$, $n/N \to \lambda_3$, where $p_1 = \int \overline{F}(u + a/\sqrt{N})\,dF(u)$ and $\overline{F}(\cdot) = 1 - F(\cdot)$.

Proof. For convenience, let us define a sequence of fragmentary samples $(X_{1,m}, Y_{1,m})', \cdots, (X_{n_m,m}, Y_{n_m,m})'$; $X_{n_m+1,m}, \cdots, X_{n_m+s_m,m}$; $Y_{n_m+1,m}, \cdots, Y_{n_m+t_m,m}$, where $(X_{i,m}, Y_{i,m} + a/\sqrt{N_m})'$ has distribution function H, $(i = 1, \cdots, n_m$, $N_m = 2n_m + s_m + t_m)$, $X_{i,m}$ and $Y_{j,m}$ have distribution functions $F(x)$ and $F(y + a/\sqrt{N_m})$, $i = 1, \cdots, n_m + s_m$, $j = 1, \cdots, n_m + t_m$, respectively, and $s_m/N_m \to \lambda_1$, $t_m/N_m \to \lambda_2$, $n_m/N_m \to \lambda_3$, as $m \to \infty$. Furthermore, we assume that $(X_{i,m}, Y_{i,m})'$, $X_{n_m+j,m}$, and $Y_{n_m+k,m}$ are mutually independent for $i = 1, \cdots, n_m$, $j = 1, \cdots, s_m$, $k = 1, \cdots, t_m$ and $(X_{1,m}, Y_{1,m})' = (U, V - a/\sqrt{N_m})'$, $m \ge 1$, where $(U,V)'$ has distribution function H.

Now, let $p_{1,m} = \int \overline{F}(u + a/\sqrt{N_m})\,dF(u)$,

$$W_m^* = \left((n_m+s_m)(n_m+t_m)N_m\right)^{-\frac{1}{2}} \left(\sum_{i=1}^{n_m+s_m} \sum_{j=1}^{n_m+t_m} \left(\phi(X_{i,m}, Y_{j,m}) - p_{1,m}\right)\right),$$

$$T_m^* = \left(\frac{n_m+t_m}{(n_m+s_m)N_m}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n_m+s_m} \left(\overline{F}(X_{i,m} + a/\sqrt{N_m}) - p_{1,m}\right)\right) +$$

$$\left(\frac{n_m+s_m}{(n_m+t_m)N_m}\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n_m+t_m} \left(F(Y_{j,m}) - p_{1,m}\right)\right), \text{ and}$$

$$g(X_{i,m}, Y_{j,m}) = \left(\phi(X_{i,m}, Y_{j,m}) - p_{1,m}\right) - \left(\overline{F}(X_{i,m} + a/\sqrt{N_m}) - p_{1,m}\right) - \left(F(Y_{j,m}) - p_{1,m}\right).$$

Then, it follows from the argument provided by Hollander, Pledger and Lin (1974, p. 179) that

$$\left((n_m+s_m)(n_m+t_m)N_m\right)^{-1} E\left(\sum_{i=1}^{n_m+s_m} \sum_{j=1}^{n_m+t_m} g(X_{i,m},Y_{j,m})\right)^2 \to 0 ,$$

as $m \to \infty$. Thus, W_m^* and T_m^* have the same limiting distribution.

To show that T_m^* is asymptotically normal, we rewrite T_m^* as the sum of three independent, normalized sums of bounded random variables:

$$\begin{aligned} T_m^* = & \left\{\left[\frac{n_m+t_m}{(n_m+s_m)N_m}\right]^{\frac{1}{2}} \sum_{i=1}^{n_m} \left(\bar{F}(X_{i,m}+a/\sqrt{N_m}) - p_{1,m}\right) + \left[\frac{n_m+s_m}{(n_m+t_m)N_m}\right]^{\frac{1}{2}} \right. \\ & \left. \sum_{i=1}^{n_m} \left(F(Y_{i,m}) - p_{1,m}\right)\right\} + \left\{\left[\frac{n_m+t_m}{(n_m+s_m)N_m}\right]^{\frac{1}{2}} \sum_{i=n_m+1}^{n_m+s_m} \left(\bar{F}(X_{i,m}+a/\sqrt{N_m}) - p_{1,m}\right)\right\} \\ & + \left\{\left[\frac{n_m+s_m}{(n_m+t_m)N_m}\right]^{\frac{1}{2}} \sum_{j=n_m+1}^{n_m+t_m} \left(F(Y_{j,m}) - p_{1,m}\right)\right\} . \end{aligned}$$

The asymptotic normality of each separate sum in T_m^* follows from a version of Berry-Esséen Theorem (c.f. Chung (1968), Theorem 7.1.2., p. 185). The asymptotic variance of T_m^* can be obtained through the fact that $(X_{1,m},Y_{1,m})' = (U,V-a/\sqrt{N_m})'$, $m \geq 1$ and $(U,V)'$ has distribution H .

Proof of Theorem 3. By Lemma 1, for any real a , $P_\theta(N^{\frac{1}{2}}(\hat{\theta}-\theta) < a) = P_0(N^{\frac{1}{2}}\hat{\theta} < a) = P_0(\hat{\theta} < a/\sqrt{N})$. Without loss of generality, we can assume that $\theta = 0$. Consider first the case $(n+s)(n+t) = 2\ell+1$. By Lemmas 2 and 3 and the fact that $\int f^2(v)dv < \infty$ (c.f. Olshen (1967) and Mehra & Sarangi (1967)),

$$P_0(D_{(\ell)} < a/\sqrt{N}) = P_0\left(W_{X,Y} - a/\sqrt{N} \le \frac{(n+s)(n+t)+1}{2}\right)$$

$$\doteq P_0\left\{\left((n+s)(n+t)N\right)^{-\frac{1}{2}}\left(W_{X,Y} - a/\sqrt{N} - (n+s)(n+t)p_1\right) \le \frac{[(n+s)(n+t)]^{\frac{1}{2}}}{N}\left[N^{\frac{1}{2}}\left(\frac{1}{2} - \int \bar{F}(u + a/\sqrt{N})\,dF(u)\right)\right]\right\} \longrightarrow \Phi\left(a(\lambda_3+\lambda_1)^{\frac{1}{2}}(\lambda_3+\lambda_2)^{\frac{1}{2}} \int f^2(v)\,dv/\eta\right) ,$$

as $N \to \infty$, $s/N \to \lambda_1$, $t/N \to \lambda_2$, $n/N \to \lambda_3$, where Φ is the distribution function of $N(0,1)$.

In the case $(n+s)(n+t) = 2\ell$, the probability $P_0(\hat{\theta} \le a/\sqrt{N})$ is bounded below and above by $P_0(D_{(\ell+1)} \le a/\sqrt{N})$ and $P_0(D_{(\ell)} < a/\sqrt{N})$. By the same argument, it can be shown that these two probabilities have the same limiting value $\Phi\left(a(\lambda_3+\lambda_1)^{\frac{1}{2}}(\lambda_3+\lambda_2)^{\frac{1}{2}} \int f^2(v)\,dv/\eta\right)$.

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TABLE 1. THE ASYMPTOTICAL RELATIVE EFFICIENCY e_i, $i = 1,2,3$.

($\lambda_1 = .1$ $\lambda_2 = .1$ $\lambda_3 = .4$)

	THE DISTRIBUTION H								
Correlation Coefficient	Bivariate normal			Bivariate logistic			Bivariate exponential		
ρ	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
-.8	.91	.91	.96	1.00	1.00	1.05	3.26	3.26	3.45
-.6	.93	.93	.96	1.03	1.03	1.06	3.26	3.25	3.36
-.4	.95	.95	.96	1.05	1.06	1.07	3.21	3.22	3.26
-.2	.96	.96	.96	1.08	1.08	1.08	3.13	3.14	3.14
0.0	.96	.96	.95	1.10	1.10	1.10	3.00	3.02	3.00
.2	.94	.95	.95	1.11	1.12	1.12	2.80	2.82	2.82
.4	.90	.91	.94	1.10	1.11	1.15	2.49	2.51	2.59
.6	.81	.82	.93	1.05	1.06	1.20	2.01	2.02	2.29
.8	.61	.62	.93	.88	.88	1.32	1.25	1.25	1.88

TABLE 2. THE ASYMPTOTICAL RELATIVE EFFICIENCY e_i, $i = 1,2,3$.

($\lambda_1 = .2$ $\lambda_2 = .2$ $\lambda_3 = .3$)

THE DISTRIBUTION H

Correlation Coefficient	Bivariate normal			Bivariate logistic			Bivariate exponential		
ρ	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
-.8	.88	.89	.96	.98	.98	1.06	3.10	3.11	3.36
-.6	.91	.93	.96	1.01	1.03	1.07	3.12	3.17	3.29
-.4	.94	.96	.96	1.05	1.08	1.08	3.12	3.21	3.21
-.2	.95	.99	.96	1.08	1.12	1.09	3.09	3.22	3.11
0.0	.95	1.01	.95	1.10	1.16	1.10	3.00	3.16	3.00
.2	.94	1.00	.95	1.10	1.17	1.11	2.84	3.01	2.87
.4	.89	.94	.94	1.06	1.13	1.13	2.55	2.71	2.71
.6	.77	.82	.94	.95	1.01	1.16	3.08	2.19	2.53
.8	.53	.55	.94	.68	.71	1.20	1.30	1.34	2.29

REFERENCES

Anderson, T. W. (1957). Maximum likelihood estimates for a multivariate normal distribution when some observations are missing, J. Amer. Statist. Assoc. 52, 200-204.

Chung, K. L. (1968). A Course in Probability Theory. Harcourt, N.Y.

Garcia-Quintana, R. A. and Huynh, H. (1980). Setting passing scores in the readiness test for beginning first graders in the State of South Carolina: Some practical considerations. Presented at the annual meeting of the Amer. Educational Research Association, Boston, Mass.

Gumbel, E. J. (1960). Bivariate exponential distributions, J. Amer. Statist. Assoc. 55, 698-707.

Gumbel, E. J. (1961). Bivariate logistic distributions, J. Amer. Statist. Assoc. 56, 335-349.

Gupta, A. K. and Rohatgi, V. K. (1979). Inference on the difference of means of correlated variables from fragmentary samples Sankhyā, Series A, to appear.

Hollander, M., Pledger, G., and Lin, P. E. (1974). Robustness of the Wilcoxon test to a certain dependency between samples, Annals of Statist. 2, 177-181.

Hocking, R. R. and Smith, W. B. (1968). Estimation of parameters in the multivariate normal distribution with missing observations, J. Amer. Statist. Assoc. 63, 159-173.

Johnson, N. L. and Kotz, S. (1976). Distributions in Statistics: Continuous Multivariate Distributions. John Wiley & Sons, Inc., New York.

Lehmann, E. L. (1975). Nonparametrics: Statistical Methods Based on Ranks. Holden-Day, Inc., San Francisco.

Lin, P. E. (1971). Estimation procedures for difference of means with missing data, J. Amer. Statist. Assoc. 66, 634-636.

Lin, P. E. (1973). Procedures for testing the difference of means with incomplete data, J. Amer. Statist. Assoc. 68, 699-703.

Lin, P. E. and Stivers, L. E. (1974). On difference of means with incomplete data, Biometrika 61, 325-334.

Mehra, K. L. and Sarangi, J. (1967). Asymptotic efficiency of certain rank tests for comparative experiments, Ann. Math. Statist. 38, 90-107.

Mehta, J. S. and Garland (1969). Some properties and an application of a statistic arising in testing correlation, Ann. Math. Statist. 40, 1736-1745.

Olshen, R. A. (1967). Sign and Wilcoxon tests for linearity, Ann. Math. Statist. 38, 1759-1769.

Randels, R. H. and Wolfe, D. A. (1979). Introduction to the Theory of Nonparametric Statistics. John Wiley & Sons, Inc., New York.